N-85620

[redacted]

N65-82308
(ACCESSION NUMBER)
23
(PAGES)
CR 60987
(NASA CR OR TMX OR AD NUMBER)
FACILITY FORM 602

(THRU)
none
(CODE)
(CATEGORY)

JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

National Aeronautics and Space Administration
Contract No. NASw-6

Technical Release No. 34-99

LOW-ACCELERATION TAKEOFF FROM A SATELLITE ORBIT

H. Lass
J. Lorell

JET PROPULSION LABORATORY
A Research Facility of
National Aeronautics and Space Administration
Operated by
California Institute of Technology
Pasadena, California
August 1, 1960

CONTENTS

I. Introduction 2

II. The Kryloff-Bogoliuboff Method 2

III. Planetary Motion with a Constant Radial Perturbing Force 3

IV. Planetary Motion Under a Constant Transverse Perturbing Force 6

V. Satellite Motion Under an Intermittent Perturbing Force 11

VI. Comparison with Other Results 17

References 21

LOW-ACCELERATION TAKEOFF FROM A SATELLITE ORBIT[1]

H. Lass[2]
J. Lorell[3]

JET PROPULSION LABORATORY
California Institute of Technology
Pasadena, California

ABSTRACT

The method of Kryloff and Bogoliuboff for handling problems in non-linear mechanics is applied to the problem of takeoff from a satellite orbit. The analysis is restricted to low acceleration, and covers three cases: constant radial thrust, constant tangential thrust, and intermittent thrust. The results are compared with those appearing elsewhere in the literature.

[1]This paper presents results of one phase of research carried out at the Jet Propulsion Laboratory, California Institute of Technology, under Contract No. NASw-6, sponsored by the National Aeronautics and Space Administration.

[2]Research Specialist, Space Sciences Division.

[3]Research Specialist, Systems Division; member, ARS.

I. INTRODUCTION

In a 1953 paper (1)[4] titled "Take-off from a Satellite Orbit," H. S. Tsien treated the equations of motion of a powered rocket in a central gravitational field. His method of analysis was direct, and led to elliptic integrals for the radial-thrust case and to series approximations for the circumferential-thrust case. Most of the subsequent literature on low-thrust trajectories continues to use Tsien's results and his direct method of approach.

Thus, Dobrowolski (2) makes use of the elliptic integral solution to get formulas for the rate of precession of the line of apses. Copeland (3) also derives the elliptic integral expressions, and in addition exhibits graphs of particular trajectories. Perkins (4), treating the case of low-level tangential thrust, uses the method of linear perturbations.

Many of the results obtained by these direct methods can be obtained much more quickly by the nonlinear techniques of Kryloff and Bogoliuboff (5). A brief statement of this method is given below. For more details, the reader is referred to Minorsky (6).

In the following Sections, the Kryloff-Bogoliuboff method is applied to three different examples of low-acceleration orbits: radial thrust, circumferential thrust, and intermittent thrust. It is seen that the results agree with those in the literature, and in certain cases are arrived at much more quickly.

II. THE KRYLOFF-BOGOLIUBOFF METHOD

An approximate solution to the nonlinear differential equation

$$\frac{dx}{dt} = \mu f(x, \sin t) \qquad [1]$$

can be obtained as follows. Equation [1] yields

$$\frac{x(t + 2\pi) - x(t)}{2\pi} = \frac{\mu}{2\pi} \int_t^{t+2\pi} f(x, \sin \tau)\, d\tau \qquad [2]$$

[4]Numbers in parentheses indicate References at end of paper.

Since μ is very small, one may consider that x essentially remains constant during the integration from $\tau = t$ to $\tau = t + 2\pi$. Since sin t is periodic, Eq. [2] becomes

$$\frac{x(t + 2\pi) - x(t)}{2\pi} \sim \frac{\mu}{2\pi} \int_0^{2\pi} f(x, \sin \tau)\, d\tau = \mu F(x) \qquad [3]$$

Finally, the left side of Eq. [3] is replaced by dx/dt, since the slope of the secant line is approximately the slope of the tangent line for $\mu << 1$. Equation [1] is replaced by

$$\frac{dx}{dt} = \mu F(x) = \frac{\mu}{2\pi} \int_0^{2\pi} f(x, \sin \tau)\, d\tau \qquad [4]$$

and Eq. [4] can be integrated by a separation of variables. Equation [4] can be obtained immediately from Eq. [1] simply by averaging the right-hand side of Eq. [1] over one cycle of the motion in the time domain. This Kryloff-Bogoliuboff method can be applied to a system of differential equations involving slowly varying quantities.

III. PLANETARY MOTION WITH A CONSTANT RADIAL PERTURBING FORCE

Consider the case of a point mass m under the influence of a central inverse-square-law force of attraction upon which is superimposed a small constant radial force ϵ. Using polar coordinates (r, θ) with origin at the center of attraction, the equations of motion are

$$\left.\begin{aligned} &\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{GM}{r^2} + \frac{\epsilon}{m}, \qquad \epsilon << 1 \\ &\frac{d}{dt}\left(r^2 \frac{d\theta}{dt}\right) = 0 \end{aligned}\right\} \qquad [5]$$

In the usual way one eliminates dt, using $r^2 d\theta = h dt$, and replaces r by $1/u$ to obtain

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{h^2} - \frac{\epsilon}{mh^2u^2} \tag{6}$$

or the system

$$\left.\begin{aligned} \frac{du}{d\theta} &= v \\ \frac{dv}{d\theta} &= -u + \frac{GM}{h^2} - \frac{\epsilon}{mh^2u^2} \end{aligned}\right\} \tag{7}$$

For $\epsilon = 0$, the solution of Eq. [7] is[5]

$$\left.\begin{aligned} u &= s_0 \sin (\theta + \theta_0) + \frac{GM}{h^2} \\ v &= s_0 \cos (\theta + \theta_0) \end{aligned}\right\} \tag{8}$$

with s_0 and θ_0 constants of integration. In order to apply the averaging process of Kryloff-Bogoliuboff, let $u = s \sin (\theta + \phi) + GM/h^2$ and $v = s \cos (\theta + \phi)$, with s and ϕ unknown functions of θ. Thus,

$$\begin{aligned} \frac{du}{d\theta} &= \frac{ds}{d\theta} \sin (\theta + \phi) + s \left(1 + \frac{d\phi}{d\theta}\right) \cos (\theta + \phi) \\ \frac{dv}{d\theta} &= \frac{ds}{d\theta} \cos (\theta + \phi) - s \left(1 + \frac{d\phi}{d\theta}\right) \sin (\theta + \phi) \end{aligned} \tag{9}$$

[5]The unperturbed solution is restricted to closed (i.e. elliptic) orbits. Hyperbolic orbits, being non-cyclic, cannot be handled by the present technique.

Equations [7] yield

$$\left.\begin{aligned}\frac{ds}{d\theta} &= -\frac{\epsilon \cos(\theta+\phi)}{mh^2\left[s \sin(\theta+\phi) + \frac{GM}{h^2}\right]^2}\\[2em] \frac{d\phi}{d\theta} &= \frac{\epsilon \sin(\theta+\phi)}{mh^2 s\left[s \sin(\theta+\phi) + \frac{GM}{h^2}\right]^2}\end{aligned}\right\} \quad [10]$$

Applying the averaging process of Kryloff-Bogoliuboff to Eq. [10] yields

$$\left.\begin{aligned}\frac{ds}{d\theta} &= -\frac{\epsilon}{2\pi mh^2}\int_0^{2\pi}\frac{\cos\tau\, d\tau}{\left(s \sin\tau + \frac{GM}{h^2}\right)^2} = 0\\[2em] \frac{d\phi}{d\theta} &= \frac{\epsilon}{2\pi mh^2 s}\int_0^{2\pi}\frac{\sin\tau\, d\tau}{\left(s \sin\tau + \frac{GM}{h^2}\right)^2} = -\frac{\epsilon}{mh^2}\left(\frac{G^2M^2}{h^4} - s^2\right)^{-\frac{3}{2}}\end{aligned}\right\} \quad [11]$$

An integration of Eq. [11] yields

$$\left.\begin{aligned}s &= s_0 = \text{constant}\\[1em] \phi &= \frac{\epsilon}{mh^2}\left(\frac{G^2M^2}{h^4} - s_0^2\right)^{-\frac{3}{2}}\theta + \phi_0\end{aligned}\right\} \quad [12]$$

so that

$$u = \frac{1}{r} = \frac{GM}{h^2} + s_0 \sin\left\{\left[1 - \frac{\epsilon}{mh^2}\left(\frac{G^2M^2}{h^4} - s_0^2\right)^{-\frac{3}{2}}\right]\theta + \phi_0\right\} \qquad [13]$$

It is seen that the line of apsides advances by the amount

$$\frac{2\pi\epsilon}{mh^2}\left(\frac{G^2M^2}{h^4} - s_0^2\right)^{-\frac{3}{2}} \text{ radians/revolution}$$

while the apogee and perigee distances remain invariant. In the case of a nearly circular orbit, $s_0 << GM/h^2$, the precession rate is $2\pi\eta$, where η is the ratio of thrust acceleration, ϵ/m, to the gravitational acceleration, GM/r^2.

IV. PLANETARY MOTION UNDER A CONSTANT TRANSVERSE PERTURBING FORCE

A properly oriented reflecting sail will experience a small transverse thrust as a result of solar radiation. In the following example, the transverse thrust is assumed to be constant.

The equations of motion are given by

$$\left.\begin{aligned} &\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{GM}{r^2} \\ &\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = \frac{\epsilon}{m}, \qquad \epsilon << 1 \end{aligned}\right\} \qquad [14]$$

Let $h = r^2\,(d\theta/dt)$, so that $dh/dt = (\epsilon/m)\,r$ and $dt = (r^2/h)\,d\theta = d\theta/hu^2$ for $u = 1/r$. Then

$$\left.\begin{aligned}
\frac{dh}{d\theta} &= \frac{dh}{dt}\,\frac{dt}{d\theta} = \frac{\epsilon}{mhu^3} \\[2ex]
\frac{dr}{dt} &= \frac{dr}{d\theta}\,\frac{d\theta}{dt} = \frac{h}{r^2}\,\frac{dr}{d\theta} = -h\,\frac{du}{d\theta} \\[2ex]
\frac{d^2r}{dt^2} &= -h^2u^2\,\frac{d^2u}{d\theta^2} - \frac{\epsilon}{mu}\,\frac{du}{d\theta}
\end{aligned}\right\} \qquad [15]$$

Letting $k = h^2$ and $v = du/d\theta$ yields

$$\left.\begin{aligned}
\frac{dk}{d\theta} &= \frac{2\epsilon}{mh^3} \\[2ex]
\frac{du}{d\theta} &= v \\[2ex]
\frac{dv}{d\theta} &= -u + \frac{GM}{k} - \frac{\epsilon v}{mku^3}
\end{aligned}\right\} \qquad [16]$$

For $\epsilon = 0$, the solutions of Eq. [16] are[5]

$$\left.\begin{aligned}
k &= k_0 = \text{constant} \\
u &= \frac{GM}{k_0} + s_0 \sin(\theta + \theta_0) \\
v &= s_0 \cos(\theta + \theta_0)
\end{aligned}\right\} \qquad [17]$$

Now let

$$\left.\begin{aligned} u &= \frac{GM}{k} + s \sin(\theta + \phi) \\ v &= s \cos(\theta + \phi) \end{aligned}\right\} \qquad [18]$$

with s, ϕ, and k unknown functions of θ. Equations [16] become

$$\left.\begin{aligned} \frac{dk}{d\theta} &= \frac{2\epsilon}{m}\left[\frac{GM}{k} + s \sin(\theta + \phi)\right]^{-3} \\ \frac{ds}{d\theta} &= \left[\frac{2\epsilon\, GM \sin(\theta + \phi)}{mk^2} - \frac{\epsilon\, s \cos^2(\theta + \phi)}{mk}\right]\left[\frac{GM}{k} + s \sin(\theta + \phi)\right]^{-3} \\ \frac{d\phi}{d\theta} &= \left[\frac{2\epsilon\, GM \cos(\theta + \phi)}{mk^2 s} + \frac{\epsilon \cos(\theta + \phi) \sin(\theta + \phi)}{mk}\right]\left[\frac{GM}{k} + s \sin(\theta + \phi)\right]^{-3} \end{aligned}\right\} \qquad [19]$$

Applying the averaging process of Kryloff-Bogoliuboff yields

$$\left.\begin{aligned} \frac{dk}{d\theta} &= \frac{\epsilon}{m} \frac{\left(\dfrac{2G^2M^2}{k^2} + s^2\right)}{\left(\dfrac{G^2M^2}{k^2} - s^2\right)^{\frac{5}{2}}} \\ \frac{ds}{d\theta} &= -\frac{\epsilon}{2mk} \frac{\left(\dfrac{7G^2M^2}{k^2} - s^2\right)}{\left(\dfrac{G^2M^2}{k^2} - s^2\right)^{\frac{5}{2}}} \end{aligned}\right\} \qquad [20]$$

$$\frac{d\phi}{d\theta} = 0 \qquad [20 \text{ cont'd}]$$

From Eq. [20], it follows that

$$\frac{ds}{dk} = \frac{s}{2k} \frac{\left(\dfrac{7G^2M^2}{k^2} - s^2\right)}{\left(\dfrac{2G^2M^2}{k^2} + s^2\right)} \qquad [21]$$

The substitution $s = w(k)/k$ makes possible the integration of Eq. [21], with the result that

$$k = K \frac{G^2M^2 - (sk)^2}{(sk)^{\frac{4}{3}}} \qquad [22]$$

with K a constant of integration which depends on the initial values of s and $k = h^2$. Equation [22] is a cubic equation in $(sk)^2$, and the algebraic solution of this equation yields $s = s(k)$.

From Eq. [20] for $\epsilon < 0$, it follows that $dk/d\theta < 0$, so that k tends to zero, and hence $r \to 0$, an expected result. From Eq. [22], it is seen that the square of the eccentricity of the orbit, $E^2 = s^2k^2/G^2M^2 \to 1$ for $\epsilon < 0$. For $\epsilon > 0$ it can be seen that $k \to \infty$ and $s \to 0$, so that $r \to \infty$.

The apogee distance is given by

$$r_a = \frac{1}{\dfrac{GM}{k} - s}, \qquad s > 0 \qquad [23]$$

so that in one cycle r_a changes by an amount Δr_a, given by

$$\Delta r_a \simeq \frac{2\pi}{\left(\dfrac{GM}{k} - s\right)^2} \left(\frac{GM}{k^2}\frac{dk}{d\theta} + \frac{ds}{d\theta}\right)$$

$$\simeq \frac{\pi\epsilon\,(4GM + sk)}{mk^2\left(\dfrac{G^2M^2}{k^2} - s^2\right)^{\frac{5}{2}}} \qquad [24]$$

For a nearly circular orbit, $r_a \simeq k/GM$, $s << 1$, it follows that

$$\Delta r_a \simeq \frac{4\pi\epsilon\, r_a^3}{GmM} = 4\pi r_a \left[\frac{\left(\dfrac{\epsilon}{m}\right)}{\left(\dfrac{GM}{r_a^2}\right)}\right] \qquad [25]$$

V. SATELLITE MOTION UNDER AN INTERMITTENT PERTURBING FORCE

It is of interest to compute the motion of a satellite of the Earth acted upon intermittently by a uniform thrust (e.g., an ion motor). The uniform thrust acts upon the satellite for the half range $0 \leqq \theta \leqq \pi$; in the half range $\pi < \theta < 2\pi$, the ion motor is shut off. This periodic thrust program is continued, and the motion of the satellite is compared with the motion of a satellite having no thrust program. The latter satellite is assumed to move in a circular path. The oblateness of the Earth is neglected since the small force field due to the Earth's oblateness affects the motion of both satellites in much the same fashion.

In polar coordinates the motion of the satellite is given by

$$\left.\begin{aligned} \frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 &= -\frac{GM}{r^2} + \mu g F \sin\theta \\ \frac{1}{r}\frac{d}{dt}\left(r^2 \frac{d\theta}{dt}\right) &= \mu g F \cos\theta \end{aligned}\right\} \qquad [26]$$

with $F = 1$ for $0 \leqq \theta \leqq \pi$, $F = 0$ for $\pi < \theta < 2\pi$. The potential per unit mass of the ion thrust motor is given by $\phi = -\mu g F y = -\mu g r \sin\theta F$, $\mu << 1$, g being the acceleration of gravity at the Earth's surface. The loss of mass of the satellite has been neglected.

Let $r^2(d\theta/dt) = h \neq$ constant, $u = 1/r$, so that

$$\left.\begin{aligned} \frac{dr}{dt} &= \frac{dr}{d\theta}\frac{h}{r^2} = -h\frac{du}{d\theta} \\ \frac{d^2 r}{dt^2} &= -h^2u^2\frac{d^2u}{d\theta^2} - \frac{\mu g}{u}\frac{du}{d\theta} F\cos\theta \\ &\cdot\ \cdot\ \cdot\ \cdot \end{aligned}\right\} \qquad [27]$$

The equations of motion become

$$\left.\begin{aligned} \frac{du}{d\theta} &= v \\ \frac{dv}{d\theta} &= -u + \frac{GM}{h^2} - \frac{\mu g}{h^2 u^2} F \sin\theta - \frac{\mu g v}{h^2 u^3} F \cos\theta \\ \frac{dh}{d\theta} &= \frac{\mu g}{h u^3} F \cos\theta \end{aligned}\right\} \quad [28]$$

The substitution

$$\left.\begin{aligned} u &= \frac{GM}{h^2} + A \sin\theta + B \cos\theta \\ v &= A \cos\theta - B \sin\theta \end{aligned}\right\} \quad [29]$$

with A and B unknown functions of θ, reduces Eq. [28] to

$$\begin{aligned} \frac{dA}{d\theta} &= \frac{2\mu g GM}{h^4 u^3} F \sin\theta \cos\theta - \frac{\mu g}{h^2 u^2} F \sin\theta \cos\theta - \frac{\mu g v}{h^2 u^3} F \cos^2\theta \\ \frac{dB}{d\theta} &= \frac{2\mu g GM}{h^4 u^3} F \cos^2\theta + \frac{\mu g}{h^2 u^2} F \sin^2\theta + \frac{\mu g v}{h^2 u^3} F \sin\theta \cos\theta \\ \frac{dh}{d\theta} &= \frac{\mu g}{h u^3} F \cos\theta \end{aligned} \quad [30]$$

The instantaneous eccentricity of the orbit is given by

$$\epsilon = \frac{h^2}{GM}\sqrt{A^2 + B^2}$$

The ϵ^2 terms, etc., are neglected so that $A^2 + B^2 << G^2M^2/h^4$. Thus

$$\left.\begin{aligned}
u &= \frac{GM}{h^2}\left(1 + \frac{Ah^2}{GM}\sin\theta + \frac{Bh^2}{GM}\cos\theta\right)\\
u^{-2} &\approx \frac{h^4}{G^2M^2}\left(1 - \frac{2Ah^2}{GM}\sin\theta - \frac{2Bh^2}{GM}\cos\theta\right)\\
u^{-3} &\approx \frac{h^6}{G^3M^3}\left(1 - \frac{3Ah^2}{GM}\sin\theta - \frac{3Bh^2}{GM}\cos\theta\right)
\end{aligned}\right\} \quad [31]$$

In order to apply the averaging process of Kryloff-Bogoliuboff to the system of equations [30] the following integrals are evaluated.

$$\left.\begin{aligned}
&\int_0^{2\pi} Fu^{-3}\sin\theta\cos\theta\, d\theta = \frac{h^6}{G^3M^3}\int_0^{\pi}\left(1 - \frac{3Ah^2}{GM}\sin\theta - \frac{3Bh^2}{GM}\cos\theta\right)\sin\theta\cos\theta\, d\theta = -\frac{2h^8B}{G^4M^4}\\
&\int_0^{2\pi} Fu^{-2}\sin\theta\cos\theta\, d\theta = \frac{h^4}{G^2M^2}\int_0^{\pi}\left(1 - \frac{2Ah^2}{GM}\sin\theta - \frac{2Bh^2}{GM}\cos\theta\right)\sin\theta\cos\theta\, d\theta = -\frac{4h^6B}{3G^2M^3}\\
&\int_0^{2\pi} Fvu^{-3}\cos^2\theta\, d\theta = \frac{h^6}{G^3M^3}\int_0^{\pi}(A\cos\theta - B\sin\theta)\cos^2\theta\, d\theta = -\frac{2h^6B}{3G^3M^3}, \text{ since } \epsilon^2 << 1\\
&\int_0^{2\pi} Fu^{-3}\cos^2\theta\, d\theta = \frac{h^6}{G^3M^3}\int_0^{\pi}\left(1 - \frac{3Ah^2}{GM}\sin\theta - \frac{3Bh^2}{GM}\cos\theta\right)\cos^2\theta\, d\theta\\
&\qquad = \frac{h^6}{G^3M^3}\left(\frac{\pi}{2} - \frac{2Ah^2}{GM}\right)
\end{aligned}\right\} \quad [32]$$

$$
\left.
\begin{aligned}
&\int_0^{2\pi} Fu^{-2}\sin^2\theta\,d\theta = \frac{h^4}{G^2M^2}\int_0^{\pi}\left(1-\frac{2Ah^2}{GM}\sin\theta-\frac{2Bh^2}{GM}\cos\theta\right)\sin^2\theta\,d\theta \\
&\qquad = \frac{h^4}{G^2M^2}\left(\frac{\pi}{2}-\frac{8Ah^2}{3GM}\right) \\
&\int_0^{2\pi} Fvu^{-3}\sin\theta\cos\theta\,d\theta = \frac{h^6}{G^3M^3}\int_0^{\pi}(A\cos\theta-B\sin\theta)\sin\theta\cos\theta\,d\theta \\
&\qquad = \frac{2h^6A}{3G^3M^3}, \text{ since } \epsilon^2 << 1 \\
&\int_0^{2\pi} Fu^{-3}\cos\theta\,d\theta = \frac{h^6}{G^3M^3}\int_0^{\pi}\left(1-\frac{3Ah^2}{GM}\sin\theta-\frac{3Bh^2}{GM}\cos\theta\right)\cos\theta\,d\theta = -\frac{3\pi h^8B}{2G^4M^4}
\end{aligned}
\right\}
\quad [32 \text{ cont'd}]
$$

Equations [30] become

$$
\begin{aligned}
\left\langle\frac{dA}{d\theta}\right\rangle &= -\frac{\mu g h^4}{\pi G^3M^3}B \\
\left\langle\frac{dB}{d\theta}\right\rangle &= \frac{3}{4}\frac{\mu g h^2}{G^2M^2}-\frac{3}{\pi}\frac{\mu g h^4}{G^3M^3}A \\
\left\langle\frac{dh}{d\theta}\right\rangle &= -\frac{3}{4}\frac{\mu g h^7}{G^4M^4}B
\end{aligned}
\qquad [33]
$$

Next, we remove the averaging symbol $\langle\,\rangle$, and integrate the system of equations [33]. Thus

$$
\frac{dA}{dh} = \frac{4GM}{3\pi h^3}
$$

$$A = -\frac{2GM}{3\pi h^2} + K \tag{34}$$

with K a constant of integration. From Eq. [33] and [34] are obtained

$$\begin{aligned} BdB &= -\frac{G^2M^2}{h^5}\left(1 + \frac{8}{3\pi}\right)dh + \frac{4KGM}{\pi h^3}\,dh \\ \frac{B^2}{2} &= \left(\frac{1}{4} + \frac{2}{3\pi}\right)\frac{G^2M^2}{h^4} - \frac{2KGM}{\pi h^2} + L \end{aligned} \tag{35}$$

with L a constant of integration. The assumption $A^2 + B^2 << G^2M^2/h^4$ enables one to deduce that h is a constant whose value can be obtained by setting the right side of Eq. [35] equal to zero and solving for h. This result is not surprising for orbits of small eccentricity, since on the average the torque produced by the thrust for the range $0 \leqq \theta \leqq \pi/2$ tends to cancel the torque produced by the thrust for the range $\pi/2 \lneqq \theta \leqq \pi$.

Integrating the second equation of [33], with h and A constants of the motion, yields

$$B = \frac{\mu g h^2}{G^2M^2}\left[\left(\frac{3}{4} + \frac{2}{\pi^2}\right) - \frac{3Kh^2}{\pi GM}\right]\theta + B_0 \tag{36}$$

Thus

$$u = \frac{1}{r} = \frac{GM}{h^2} + \left(K - \frac{2GM}{3\pi h^2}\right)\sin\theta + \left\{\frac{\mu g h^2}{G^2M^2}\left[\left(\frac{3}{4} + \frac{2}{\pi^2}\right) - \frac{3Kh^2}{\pi GM}\right]\theta + B_0\right\}\cos\theta \tag{37}$$

If the satellite is initially in circular motion, then $r = R$, $GM/h^2 = 1/R$, $u = 1/R$, $\dot{u} = 0$, at $\theta = 0$, $t = 0$, so that

$$\begin{aligned} K - \frac{2GM}{3\pi h^2} + \frac{\mu g h^2}{G^2M^2}\left[\left(\frac{3}{4} + \frac{2}{\pi^2}\right) - \frac{3Kh^2}{\pi GM}\right] &= 0 \\ B_0 &= 0 \end{aligned} \tag{38}$$

Hence

$$K = \frac{\dfrac{2GM}{3\pi h^2} - \dfrac{\mu g h^2}{G^2 M^2}\left(\dfrac{3}{4} + \dfrac{2}{\pi^2}\right)}{1 - \dfrac{3\mu g h^4}{\pi G^3 M^3}} \qquad [39]$$

$$K \approx \frac{2GM}{3\pi h^2}\left(1 - \frac{9\pi\mu g h^4}{8G^3 M^3}\right)$$

and

$$u = \frac{1}{r} = \frac{GM}{h^2} - \frac{3\mu g h^2}{4G^2 M^2}\sin\theta + \frac{3\mu g h^2}{4G^2 M^2}\theta\cos\theta \qquad [40]$$

neglecting μ^2 terms, so that

$$r \approx R\left[1 + \frac{3\mu g R^2}{4GM}\left(\sin\theta - \theta\cos\theta\right)\right] \qquad [41]$$

since $h^2 = GMR$. The change in r per revolution is

$$\Delta r = -\frac{\dfrac{3}{2}\pi\mu g R^3}{GM} \qquad [42]$$

The instantaneous eccentricity is given by

$$\epsilon = \frac{3}{4}\frac{\mu g R^2}{GM}\sqrt{1 + \theta^2} \qquad [43]$$

For $\mu \approx 10^{-5}$, $GM/R^2 \approx g$, one has $\epsilon \approx (3/4)(10^{-5})\sqrt{1+\theta^2}$, so that the eccentricity will remain small for a considerable number of revolutions. Thus we are justified in omitting ϵ^2 terms, etc.

From $r^2(d\theta/dt) = h$, and from Eq. [41] one obtains

$$\theta \approx \frac{h}{R^2} t + \frac{3\mu g R^2}{GM}\left[\cos\left(\frac{h}{R^2} t\right) + \frac{ht}{2R^2}\sin\left(\frac{h}{R^2} t\right) - 1\right] \tag{44}$$

The coordinates of the unperturbed satellite are $r_0 = R$, $\theta_0 = ht/R^2$. The distance ρ between the two satellites is given by

$$\rho = \frac{3\mu g R^3}{GM}\left\{\frac{1}{16}\left[\sin\left(\frac{ht}{R^2}\right)-\left(\frac{ht}{R^2}\right)\cos\left(\frac{ht}{R^2}\right)\right]^2 + \cos\left[\left(\frac{ht}{R^2}\right)+\frac{ht}{2R^2}\sin\left(\frac{ht}{R^2}\right)-1\right]^2\right\}^{\frac{1}{2}} \tag{45}$$

VI. COMPARISON WITH OTHER RESULTS

It was shown above, in the discussion of a constant radial perturbing force that for a small radial thrust the trajectory is given by Eq. [13] which may be written in the form

$$r = \frac{\dfrac{h^2}{GM}}{1 + \dfrac{s_0 h^2}{GM}\sin\left[(1-\Gamma)\,\theta + \phi_0\right]} \tag{46}$$

where

$$\Gamma = \frac{\epsilon}{mh^2}\left(\frac{G^2M^2}{h^4} - s_0^2\right)^{-\frac{3}{2}} \qquad [47]$$

Equation [1] is almost in standard form for an ellipse. In fact, by setting $\epsilon = 0$ it is seen that the osculating ellipse has the following parameters:

semilatus-rectum = h^2/GM

eccentricity = $s_0 h^2/GM$

angle from pericenter to origin = $\phi_0 \quad \pi/2$

In particular, for a near-circular orbit s_0 is small. Therefore, by Eq. [46],

$$h^2 \sim GMr \qquad [48]$$

and by Eq. [47],

$$\Gamma \sim \frac{\epsilon}{m} : \frac{GM}{r^2} = \eta \qquad [49]$$

It follows from Eq. [46] and [49] that the line of apsides advances by the amount $2\pi\eta$ radians/revolution.

Dobrowolski indicates a regression of the line of apses. However, he apparently used an incorrect expansion for his parameter b to substitute in his general solution. Substitution of the correct expansion gives a result in agreement with ours. Copeland's Fig. 2 further confirms the result that the line of apsides advances rather than regresses.

For near-circular motion we can evaluate the constants s_0 and ϕ_0 as follows. Assume that initially $r = r_0$, and $\dot{r} = 0$ and $\theta = 0$. Then, by Eq. [46],

$$\frac{1}{r_0} = \frac{GM}{h^2} + s_0 \sin \phi_0 \qquad [50]$$

and, from the first derivative of Eq. [46],

$$0 = s_0 \ (1 - \Gamma) \cos \phi_0 \tag{51}$$

It then follows that

$$\phi_0 = \frac{\pi}{2} \tag{52}$$

and

$$s_0 = \frac{GM}{h^2} - \frac{1}{r_0} \tag{53}$$

But, since we are assuming near-circular motion, Eq. [6] gives $d^2u/dG^2 \simeq 0$, and

$$\frac{GM}{h^2} - \frac{1}{r} \sim \frac{\eta}{r} \tag{54}$$

hence, substituting in Eq. [53],

$$s_0 \sim \frac{\eta}{r_0} \tag{55}$$

To get the maximum excursion of the radial distance r, it is only necessary to take the difference between r_{max} and r_{min} as obtained from Eq. [46] and write

$$\frac{\Delta r}{r_0} = \frac{\dfrac{2s_0 h^4}{r_0 G^2 M^2}}{1 - \dfrac{s_0 h^4}{G^2 M^2}} \tag{56}$$

which, for the near-circular orbit reduces to

$$\frac{\Delta r}{r_0} \sim \frac{2s_0r_0}{1 - s_0^2r_0^2} \sim \frac{2\eta}{1 - \eta^2} \sim 2\eta \qquad [57]$$

in agreement with Dobrowolski.

In the case of circumferential thrust, the maximum radial distance in each cycle does not remain stationary. Its rate of growth is given by Eq. [25] for the nearly circular orbit, i.e.

$$\frac{\Delta r_a}{r_a} \sim 4\pi\eta$$

This result agrees directly with that of Perkins (4) as can be seen by substituting $\Delta\phi = 2\pi$ in Eq. [29] of Perkin's paper.

Briefly, application of the non-linear techniques of Kryloff and Bogoliuboff has been shown to provide concise solutions to the problems of small radial thrust and small circumferential thrust previously discussed in the literature, and to the problem of intermittent thrust, not previously published. It is hoped that this work will inspire more extensive application of the method.

REFERENCES

1. Tsien, H.S., "Take-Off From Satellite Orbit", *Journal of the American Rocket Society*, 23:233-236, July-August 1953.

2. Kryloff, N., and N. Bogoliuboff. "Introduction to Non-Linear Mechanics," Kiev, 1937. Chapters 10-12. Translation: tr by Solomon Lofschetz, Princeton University Press, Princeton, 1943. (See Ref. 6).

3. Dobrowolski, A., "Satellite Orbit Perturbations Under a Continuous Radial Thrust of Small Magnitude," *Jet Propulsion*, 28:687-8, October 1958.

4. Copeland, Jack, "Interplanetary Trajectories Under Low-Thrust Radial Acceleration," *ARS Journal*, 29:267-271, April 1959.

5. Perkins, Frank M., "Flight Mechanics of Low-Thrust Spacecraft," *Journal of the Aero/Space Sciences*, 26:291-297, May 1959.

6. Minorsky, N., "Introduction to Non-Linear Mechanics", J. W. Edwards, Ann Arbor, 1947. Chapter X, based on cited chapters of Kryloff and Bogoliuboff (Ref. 2).